Solving polynomial equations

Introduction

Linear and quadratic equations, dealt within Blocks 1 and 2 are members of a class of equations called **polynomial equations**. These have the general form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

in which x is a variable and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are given constants. Also n must be a positive integer. Examples include $x^3 + 7x^2 + 3x - 2 = 0$, $5x^4 - 7x^2 = 0$ and $-x^6 + x^5 - x^4 = 0$. In this block you will learn how to factorise some polynomial expressions and solve some polynomial equations.

Prerequisites

Before starting this Block you should . . .

- be able to solve linear and quadratic equations

Learning Outcomes

After completing this Block you should be able to . . .

✓ recognise and solve some polynomial equations

Learning Style

To achieve what is expected of you . . .

☞ allocate sufficient study time

☞ briefly revise the prerequisite material

☞ attempt *every* guided exercise and most of the other exercises

1. Multiplying polynomials together

> **Key Point**
>
> A **polynomial expression** is one of the form
>
> $$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$
>
> where a_0, a_1, ..., a_n are known coefficients (numbers) and x is a variable.
> n must be a positive integer.

For example $x^3 - 17x^2 + 54x - 8$ is a polynomial expression in x. The polynomial may be expressed in terms of a variable other than x. So, the following are also polynomial expressions:

$$t^3 - t^2 + t - 3 \qquad z^5 - 1 \qquad w^4 + 10w^2 - 12$$

Note that only non-negative whole number powers of the variable x are allowed in a polynomial expression. In this block you will learn how to factorise simple polynomial expressions and how to solve some polynomial equations. You will also learn the technique of **equating coefficients**. This process is very important when we need to perform calculations involving partial fractions which will be considered in Block 6.

The **degree** of a polynomial is the highest power to which the variable is raised. Thus $x^3 + 6x + 2$ has degree 3, $t^6 - 6t^4 + 2t$ has degree 6, and $5x + 2$ has degree 1.

Let us consider what happens when two polynomials are multiplied together. For example

$$(x + 1)(3x - 2)$$

is the product of two first degree polynomials. Expanding the brackets we obtain

$$(x + 1)(3x - 2) = 3x^2 + x - 2$$

which is a second degree polynomial.

In general we can regard a second degree polynomial, or quadratic, as the product of two first degree polynomials, provided that the quadratic can be factorised.

On the other hand

$$(x - 1)(x^2 + 3x - 7) = x^3 + 2x^2 - 10x + 7$$

is a third degree, or **cubic**, polynomial which is thus the product of a linear polynomial and a quadratic polynomial.

In general we can regard a cubic polynomial as the product of a linear polynomial and a quadratic polynomial. This fact will be important in the following section when we come to factorise cubics.

Key Point

A cubic expression is a linear expression times a quadratic expression

1. If $x^3 - 17x^2 + 54x - 8 = (x - 4) \times$ (a polynomial), state the degree of the undefined polynomial.

2. (a) If $3x^2 + 13x + 4 = (x + 4) \times$ (a polynomial), state the degree of the undefined polynomial.

   (b) What is the coefficient of x in this unknown polynomial ?

3. If $2x^2 + 5x + 2 = (x + 2) \times$ (a polynomial), what must be the coefficient of x in this unknown polynomial ?

4. Two quadratic polynomials are multiplied together. What is the degree of the resulting polynomial ?

1.

Your solution

second.

2. (a)

Your solution

first

(b)

Your solution

it must be 3 in order to generate the term $3x^2$ when the brackets are removed.

3.

Your solution

it must be 2 in order to generate the term $2x^2$ when the brackets are removed.

4.

fourth degree.

2. Factorising polynomial expressions and equating coefficients

Eventually we will consider how we might find the solution to some simple polynomial equations. An important part of this process is being able to express a complicated polynomial into a product of simpler polynomials. This involves **factorisation**.

Factorisation of polynomial expressions can sometimes be achieved if one or more of the factors is already known. This requires a knowledge of the technique of 'equating coefficients' which is illustrated in the following example.

Example factorise the expression $x^3 - 17x^2 + 54x - 8$ given that one of the factors is $(x - 4)$.

Solution

Given that $x - 4$ is a factor we can write

$$x^3 - 17x^2 + 54x - 8 = (x - 4) \times \text{(a quadratic polynomial)}$$

The polynomial must be quadratic because the expression on the left is cubic. Suppose we write this quadratic as $ax^2 + bx + c$ where a, b and c are unknown numbers which we will try to find. Then

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(ax^2 + bx + c)$$

Removing the brackets on the right and collecting like terms together we have

$$x^3 - 17x^2 + 54x - 8 = ax^3 + (b - 4a)x^2 + (c - 4b)x - 4c$$

Equating coefficients means that we compare the coefficients of each term on the left with the corresponding term on the right. Thus if we look at the x^3 terms on each side we see that

$$x^3 = ax^3$$

that is a must equal 1. Similarly by equating coefficients of x^2 we find

$$-17 = b - 4a$$

With $a = 1$ we have $-17 = b - 4$ so that b must equal -13. Finally, equating constant terms we find

$$-8 = -4c$$

so that $c = 2$.

Solution

Check for yourself that with these values of c and b, the coefficient of x is the same on both sides. We can now write the polynomial expression as

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2)$$

Exercises

Factorise the given polynomial expressions

1. $x^3 - 6x^2 + 11x - 6$, given that $x - 1$ is a factor

2. $x^3 - 7x - 6$, given that $x + 2$ is a factor

3. $2x^3 + 7x^2 + 7x + 2$, given that $x + 1$ is a factor

4. $3x^3 + 7x^2 - 22x - 8$, given that $x + 4$ is a factor

Answer 1. $(x - 1)(x^2 - 5x + 6)$, 2. $(x + 2)(x^2 - 2x - 3)$, 3. $(x + 1)(2x^2 + 5x + 2)$, 4. $(x + 4)(3x^2 - 5x - 2)$.

3. Polynomial equations

When a polynomial expression is equated to zero, a polynomial equation is obtained. Linear and quadratic equations, which you have already met, are particular types of polynomial equation.

Key Point

A **polynomial equation** has the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0 = 0$$

where a_0, a_1, ..., a_n are known coefficients, and x represents an unknown whose value(s) are to be found.

Polynomial equations of low degree have special names. A polynomial equation of degree 1 is a linear equation and such equations have been solved in Block 1. Degree 2 polynomials are quadratics; degree 3 polynomials are called cubics; degree 4 equations are called quartics and so on. The following are examples of polynomial equations:

$$5x^6 - 3x^4 + x^2 + 7 = 0, \qquad -7x^4 + x^2 + 9 = 0, \qquad t^3 - t + 5 = 0, \qquad w^7 - 3w - 1 = 0$$

Recall that the **degree** of the equation is the highest power of x occurring. The **solutions** or **roots** of the equation are those values of x which satisfy the equation.

Key Point

A polynomial equation of degree n has n roots

Some (possibly all) of the roots may be repeated
Some (possibly all) of the roots may be complex.

Example Verify that $x = -1$, $x = 1$ and $x = 0$ are solutions (roots) of the equation

$$x^3 - x = 0$$

Solution

We substitute each value in turn into $x^3 - x$.

$$(-1)^3 - (-1) = -1 + 1 = 0$$

so $x = -1$ is clearly a root. It is easy to verify similarly that $x = 1$ and $x = 0$ are also solutions.

We now consider ways in which polynomial equations of higher degree can be solved.

Exercises

Verify that the given values are solutions of the given equations.

1. $x^2 - 5x + 6$, $x = 3, x = 2$

2. $2t^3 + t^2 - t$, $t = 0, t = -1, t = \frac{1}{2}$.

4. Solving polynomial equations when one solution is known

In Block 2 we gave a formula which can be used to solve quadratic equations. Unfortunately when dealing with equations of higher degree no simple formulae exist. If one of the roots can be spotted we can sometimes find the others by the method shown in the next example.

Example Let the polynomial expression $x^3 - 17x^2 + 54x - 18$ be denoted by $P(x)$. Verify that $x = 4$ is a solution of the equation $P(x) = 0$. Hence find the other solutions.

Solution

We substitute $x = 4$ into the polynomial expression $P(x)$:

$$4^3 - 17(4^2) + 54(4) - 8 = 64 - 272 + 216 - 8 = 0$$

So, when $x = 4$ the left-hand side equals zero. Hence $x = 4$ is indeed a solution. Knowing that $x = 4$ is a root **we can state that** $(x - 4)$ **must be a factor of** $P(x)$. Therefore $P(x)$ can be re-written as a product of a linear and a quadratic term:

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4) \times \text{(quadratic polynomial)}$$

The quadratic polynomial has already been found on page 4. So the given equation can be written

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2) = 0$$

In this form we see that

$$x - 4 = 0 \qquad \text{or} \qquad x^2 - 13x + 2 = 0$$

Solution

The first equation gives $x = 4$ which we already knew. The second must be solved using one of the methods for solving quadratic equations given in Block 2. For example, using the formula we find

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{with } a = 1, \ b = -13, \ c = 2 \\
&= \frac{13 \pm \sqrt{(-13)^2 - 4.1.2}}{2} \\
&= \frac{13 \pm \sqrt{161}}{2} \\
&= \frac{13 \pm 12.6886}{2}
\end{aligned}
$$

So $x = 12.844$ and $x = 0.156$ are roots of $x^2 - 13x + 2$. Hence the solutions of $P(x) = 0$ are $x = 4$, $x = 12.844$ and $x = 0.156$.

Solve the equation $x^3 + 8x^2 + 16x + 3 = 0$ given that $x = -3$ is a root.

Consider the equation $x^3 + 8x^2 + 16x + 3 = 0$. Since $x = -3$ is a root then $(x + 3)$ must be a factor of the left-hand side. We can therefore write the left-hand side as

$$x^3 + 8x^2 + 16x + 3 = (x + 3)(ax^2 + bx + c)$$

where a, b, and c are constants. These can be found by expanding the right hand side and equating coefficients of powers of x. Expanding the right-hand side we have

$$x^3 + 8x^2 + 16x + 3 = ax^3 + (3a + b)x^2 + (3b + c)x + 3c$$

Your solution

Equating coefficients of x^3 we find $a =$

1

Your solution

Equating constant terms we find

$3 = 3c$ so that $c = 1$

Your solution

Finally equating coefficients of x^2 gives

$8 = 3a + b$

from which $b = 5$.

This enables us to write the equation as

$$(x + 3)(x^2 + 5x + 1) = 0$$

Thus $x + 3 = 0$ or $x^2 + 5x + 1 = 0$.

The quadratic equation can be solved using the formula to obtain $x = -4.791$ and $x = -0.209$. Thus the three roots of $x^3 + 8x^2 + 16x + 3$ are $x = -3$, $x = -4.791$ and $x = -0.209$.

Exercises

Verify that the given value is a solution of the equation and hence find all solutions:

1. $x^3 + 7x^2 + 11x + 2 = 0, \qquad x = -2$
2. $2x^3 + 11x^2 - 2x - 35 = 0, \qquad x = -5$
3. Verify that $x = 1$ and $x = 2$ are solutions of $x^4 + 4x^3 - 17x^2 + 8x + 4$ and hence find all solutions.

Answer 1. -2, -0.2087, -4.7913 2. $-5, -2.1375, 1.6375$
3. 1,2, -0.2984, -6.7016,

5. Solving a polynomial equation graphically

Polynomial equations, particularly of high degree, are difficult to solve unless they take a particularly simple form. A useful guide to the approximate values of the solutions can be obtained by sketching the polynomial, and discovering where the curve crosses the x-axis The real roots of the polynomial equation $P(x) = 0$ are given by the values of the intercepts of the function $y = P(x)$ with the x-axis because on the x-axis, y and hence $P(x)$, is zero. Computer software packages and graphics calculators exist which can be used for plotting graphs and hence for solving polynomial equations. Suppose the graph of $y = P(x)$ is plotted and takes a form similar to that shown in Figure 1.

y
x
x_1
x_2
x_3

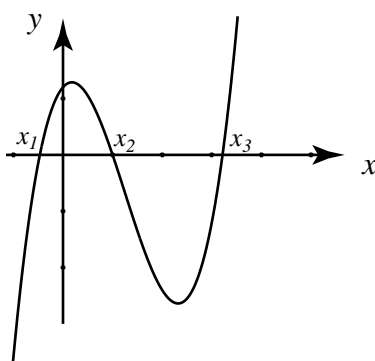

Figure 1 A polynomial function which cuts the x axis at points x_1, x_2 and x_3.

The graph intersects the x axis at $x = x_1$, $x = x_2$ and $x = x_3$ and so the equation $P(x) = 0$ has three roots x_1, x_2 and x_3.

Example Plot a graph of the function $y = 4x^4 - 15x^2 + 5x + 6$ and hence solve the equation $4x^4 - 15x^2 + 5x + 6 = 0$.

Solution

The graph has been plotted with the aid of a computer graph plotting package and is shown in Figure 2.
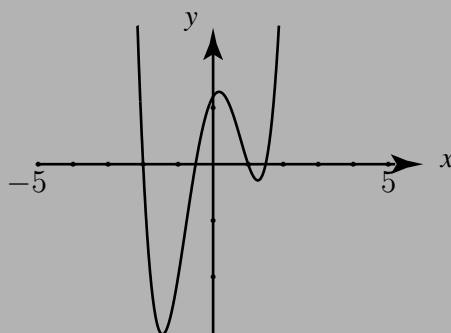

Figure 2. Graph of $y = 4x^4 - 15x^2 + 5x + 6$

The solutions of the equation are found by looking for where the graph crosses the horizontal axis. Careful examination shows that the solutions are $x = 1$, $x = 1.5$, $x = -0.5$ and $x = -2$.

An important feature of the graph of a polynomial is that it is continuous. There are never any gaps or jumps in the curve. Polynomial curves never turn back on themselves (unlike a circle) By studying the graph in Figure 2 you will see that if we choose any two values of x, say a and b, such that $y(a)$ and $y(b)$ have opposite signs, then at least one root lies between $x = a$ and $x = b$.

Exercises

1. Consider the polynomial $P(x) = 5x^3 - 47x^2 + 84x$. By evaluating $P(2)$ and $P(3)$ show that at least one root of $P(x) = 0$ lies between $x = 2$ and $x = 3$.

Exercises

1. Factorise $x^3 - x^2 - 65x - 63$ given that $(x + 7)$ is a factor.

2. Show that $x = -1$ is a root of $x^3 + 11x^2 + 31x + 21 = 0$ and locate the other roots algebraically.

3. Show that $x = 2$ is a root of $x^3 - 3x - 2 = 0$ and locate the other roots.

4. Solve the equation $x^4 - 2x^2 + 1 = 0$.

5. factorise $x^4 - 7x^3 + 3x^2 + 31x + 20$ given that $(x + 1)$ is a factor.

6. Given that two of the roots of $x^4 + 3x^3 - 7x^2 - 27x - 18 = 0$ have the same modulus but different sign, solve the equation.

   (Hint - let two of the roots be α and $-\alpha$ and use the technique of equating coefficients).

7. Without solving the equation or using a graphical calculator, show that $x^4 + 4x - 1 = 0$ has a root between $x = 0$ and $x = 1$.

Answer 1. $(x + 7)(x + 1)(x - 9)$ 2. $x = -1, -3, -7$ 3. $x = 2, -1$
4. $x = -1, 1$ 5. $(x + 1)^2(x - 4)(x - 5)$ 6. $(x + 3)(x - 3)(x + 1)(x + 2)$